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Study of Fractional Laplace Transform

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Abstract: **In this article, we study the fractional Laplace transforms of fractional analytic functions based on Jumarie type of modified Riemann-Liouville (R-L) fractional derivative. A new multiplication of fractional analytic functions plays an important role in this paper. And the results we obtained are generalizations of Laplace transforms of classical analytic functions.**

Keywords: **fractional Laplace transforms, fractional analytic functions, Jumarie type of modified R-L fractional derivative, new multiplication.**

I. INTRODUCTION

In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field. We can mention Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Hardy, Littlewood, and Weyl. In the past decades, fractional calculus has been considered as one of the best tools to describe the process of long memory. Such models are interesting for physicists, engineers, and mathematicians. Fractional calculus has important applications in various fields such as physics, mechanics, electricity, biology, economics, control theory, and so on. The introduction and application of fractional calculus can refer to [1-9]. Fractional calculus includes the derivative and integral of any real or complex order. There is no unique definition of fractional derivative and integral. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letinikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [1-4].

In this article, based on Jumarie type of modified R-L fractional derivative, we evaluate the fractional Laplace transforms of some fractional analytic functions such as fractional exponential function, fractional sine and cosine functions, and fractional hyperbolic sine and cosine functions. A new multiplication of fractional analytic functions plays an important role in this paper, and the results we obtained are natural generalizations of the results in classical Laplace transform. For the introduction and application of fractional Laplace transform can refer to [10-11]

II. DEFINITIONS AND PROPERTIES

Firstly, we introduce the fractional calculus used in this article.

Definition 2.1: If α is a real number, and m is a positive integer. The Jumarie's modified Riemann-Liouville fractional derivative [12] is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}(t-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\leq\alpha<1 \\ \frac{d^m}{dt^m}\left({}_{t_0}D_t^{\alpha-m}\right)[f(t)], & \text{if } m\leq\alpha<m+1 \end{cases} \quad (1)$$

where $\Gamma(\ )$ is the gamma function. On the other hand, we define the α-fractional integral of $f(t)$ by $\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \left({}_{t_0}D_t^{-\alpha}\right)[f(t)]$, where $\alpha>0$. If $\left({}_{t_0}I_t^{\alpha}\right)[f(t)]$ exists, then $f(t)$ is called an α-fractional integrable function. We have the following properties.

Proposition 2.2: *If α,β,c are real numbers and $\beta\geq\alpha>0$, then*

$${}_0D_t^{\alpha}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \quad (2)$$

and

$$ {}_0D_t^{\alpha}[c] = 0. \tag{3}$$

Next, we define the fractional analytic function.

Definition 2.3 ([13])**:** Assume that , t_0 , and a_k are real numbers for all k, $t_0 \in (a,b)$, and let $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval $(t_0 - r, t_0 + r)$, then we say that $f_\alpha(t^\alpha)$ is α-fractional analytic at t_0, where r is the radius of convergence about t_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, some fractional analytic functions are introduced.

Definition 2.4 ([14, 16])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{4}$$

where α is a real number, $\alpha \ge 0$, and z is a complex number.

Definition 2.5 ([15])**:** Assume that $0 < \alpha \le 1$, and p, t are real numbers. $E_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{p^k t^{k\alpha}}{\Gamma(k\alpha+1)}$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k} t^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{5}$$

and

$$sin_\alpha(pt^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k+1} t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{6}$$

Moreover, the α-fractional hyperbolic cosine function and hyperbolic sine function are defined by

$$cosh_\alpha(pt^\alpha) = \frac{1}{2}[E_\alpha(pt^\alpha) + E_\alpha(-pt^\alpha)], \tag{7}$$

$$sinh_\alpha(pt^\alpha) = \frac{1}{2}[E_\alpha(pt^\alpha) - E_\alpha(-pt^\alpha)]. \tag{8}$$

Remark 2.6: If $\alpha = 1, p = 1$, then $cos_1(t) = cost$, and $sin_1(t) = sint$.

Notation 2.7: Suppose that $z = a + ib$ is a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. Then a, the real part of z, is denoted by $\text{Re}(z)$; b, the imaginary part of z, is denoted by $\text{Im}(z)$.

Proposition 2.8 (fractional Euler's formula)**:** *Assume that* $0 < \alpha \le 1$, *then*

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha)\,. \tag{9}$$

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.9 ([13])**:** Assume that $0 < \alpha \le 1$, $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} t^{k\alpha}, \tag{10}$$

$$g_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} t^{k\alpha}. \tag{11}$$

Then we define

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} t^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} t^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) t^{k\alpha}. \end{aligned} \tag{12}$$

Theorem 2.10 (integration by parts for fractional calculus)([17])**:** *If* $0 < \alpha \leq 1$, a, b *are real numbers, and* $f_\alpha(t^\alpha), g_\alpha(t^\alpha)$ are α-fractional analytic functions, *then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(t^\alpha) \otimes \left({}_aD_t^\alpha\right)[g_\alpha(t^\alpha)]\right] = f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha)|_a^b - \left({}_aI_b^\alpha\right)\left[g_\alpha(t^\alpha) \otimes \left({}_aD_t^\alpha\right)[f_\alpha(t^\alpha)]\right]. \tag{13}$$

The following is the definition of fractional Laplace transform.

Definition 2.11: Assume that $0 < \alpha \leq 1$, s is a real variable, and $f_\alpha(t^\alpha)$ is an α-fractional analytic functions defined for all $t \geq 0$. The function $F_\alpha(s)$ defined by the α-fractional improper integral $\left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-st^\alpha) \otimes f_\alpha(t^\alpha)]$ is called the α-fractional Laplace transform of the function f_α , and is denoted by $L_\alpha\{f_\alpha(t^\alpha)\}$. That is,

$$F_\alpha(s) = L_\alpha\{f_\alpha(t^\alpha)\} = \left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-st^\alpha) \otimes f_\alpha(t^\alpha)]. \tag{14}$$

Proposition 2.12 ([15])**:** *If* $0 < \alpha \leq 1$, *and* p, q *are real number, then*

$$E_\alpha(pt^\alpha) \otimes E_\alpha(qt^\alpha) = E_\alpha((p+q)t^\alpha). \tag{15}$$

III. MAIN RESULTS

In the following, we introduce the major results in this paper.

Proposition 3.1 (linearity of fractional Laplace transform)**:** *The fractional Laplace transform is a linear operation; that is, for any fractional analytic functions* $f_\alpha(t^\alpha)$ *and* $g_\alpha(t^\alpha)$ *whose fractional Laplace transforms exist, then for any constants* a *and* b, *the fractional Laplace transform* $af_\alpha(t^\alpha) + b\,g_\alpha(t^\alpha)$ *exists and*

$$L_\alpha\{a\,f_\alpha(t^\alpha) + b\,g_\alpha(t^\alpha)\} = aL_\alpha\{f_\alpha(t^\alpha)\} + bL_\alpha\{g_\alpha(t^\alpha)\}. \tag{16}$$

Theorem 3.2 (first shifting theorem for fractional Laplace transform)**:** *Suppose that* $0 < \alpha \leq 1, p, k$ *are real numbers, and* $f_\alpha(t^\alpha)$ *has the fractional Laplace transform* $F_\alpha(s)$ *for* $s > k$. *Then* $E_\alpha(-pt^\alpha) \otimes f_\alpha(t^\alpha)$ *has the fractional Laplace transform* $F_\alpha(s-p)$ *for* $s > k+p$. *In formula,*

$$L_\alpha\{E_\alpha(pt^\alpha) \otimes f_\alpha(t^\alpha)\} = F_\alpha(s-p). \tag{17}$$

Proof Since $F_\alpha(s) = L_\alpha\{f_\alpha(t^\alpha)\} = \left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-st^\alpha) \otimes f_\alpha(t^\alpha)]$ exists for $s > k$, it follows that

$$\begin{aligned} F_\alpha(s-p) &= \left({}_0I_{+\infty}^\alpha\right)[E_\alpha(-(s-p)t^\alpha) \otimes f_\alpha(t^\alpha)] \\ &= \left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha(-st^\alpha) \otimes \left(E_\alpha(pt^\alpha) \otimes f_\alpha(t^\alpha)\right)\right] \\ &= L_\alpha\{E_\alpha(pt^\alpha) \otimes f_\alpha(t^\alpha)\} \text{ exists for } s > k+p. \end{aligned}$$

Q.e.d.

Proposition 3.3: *Let* $0 < \alpha \leq 1$, s, t, p, ω *be real numbers,* $t \geq 0$, *and* n *be a positive integer. Then*

$$L_\alpha\{1\} = \frac{1}{s}, \text{ where } s > 0 \tag{18}$$

$$L_\alpha\{E_\alpha(pt^\alpha)\} = \frac{1}{s-p}, \text{ where } s > p \tag{19}$$

$$L_\alpha\{t^{n\alpha}\} = \frac{\Gamma(n\alpha+1)}{s^{n+1}}, \text{ where } s > 0 \tag{20}$$

$$L_\alpha\{cos_\alpha(\omega t^\alpha)\} = \frac{s}{s^2+\omega^2}, \text{ where } s > 0 \tag{21}$$

$$L_\alpha\{sin_\alpha(\omega t^\alpha)\} = \frac{\omega}{s^2+\omega^2}, \text{ where } s > 0 \tag{22}$$

$$L_\alpha\{cosh_\alpha(pt^\alpha)\} = \frac{s}{s^2-p^2}, \text{ where } s > |p| \tag{23}$$

$$L_\alpha\{sinh_\alpha(pt^\alpha)\} = \frac{p}{s^2-p^2}, \text{ where } s > |p| \tag{24}$$

$$L_\alpha\{E_\alpha(pt^\alpha) \otimes cos_\alpha(\omega t^\alpha)\} = \frac{s-p}{(s-p)^2+\omega^2}, \text{ where } s > p \tag{25}$$

$$L_\alpha\{E_\alpha(pt^\alpha) \otimes sin_\alpha(\omega t^\alpha)\} = \frac{\omega}{(s-p)^2+\omega^2}, \text{ where } s > p \tag{26}$$

Proof

$$L_\alpha\{1\} = ({}_0I^\alpha_{+\infty})[E_\alpha(-st^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-st^\alpha)]$$
$$= \lim_{t\to+\infty}\left[-\frac{1}{s}E_\alpha(-st^\alpha)+\frac{1}{s}\right]$$
$$= \frac{1}{s} \text{ (since } s>0).$$

$$L_\alpha\{E_\alpha(pt^\alpha)\} = ({}_0I^\alpha_{+\infty})[E_\alpha(-st^\alpha)\otimes E_\alpha(pt^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-st^\alpha)\otimes E_\alpha(pt^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-(s-p)t^\alpha)]$$
$$= \lim_{t\to+\infty}\left[-\frac{1}{s-p}E_\alpha(-(s-p)t^\alpha)+\frac{1}{s-p}\right]$$
$$= \frac{1}{s-p} \text{ (since } s>p).$$

$$L_\alpha\{t^{n\alpha}\} = ({}_0I^\alpha_{+\infty})[E_\alpha(-st^\alpha)\otimes t^{n\alpha}]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-st^\alpha)\otimes t^{n\alpha}]$$
$$= \lim_{t\to+\infty} -\frac{1}{s}E_\alpha(-st^\alpha)\otimes t^{n\alpha}|_0^t + \frac{1}{s}\cdot\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}({}_0I^\alpha_t)\left[E_\alpha(-st^\alpha)\otimes t^{(n-1)\alpha}\right]$$

(by integration by parts for fractional calculus)

$$= \frac{1}{s}\cdot\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}({}_0I^\alpha_{+\infty})\left[E_\alpha(-st^\alpha)\otimes t^{(n-1)\alpha}\right] \text{ (since } s>0).$$

By induction, we obtain $L_\alpha\{t^{n\alpha}\} = \frac{\Gamma(n\alpha+1)}{s^{n+1}}$.

$$L_\alpha\{cos_\alpha(\omega t^\alpha)\} = ({}_0I^\alpha_{+\infty})[E_\alpha(-st^\alpha)\otimes cos_\alpha(\omega t^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-st^\alpha)\otimes cos_\alpha(\omega t^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)\left[\text{Re}[E_\alpha((-s+i\omega)t^\alpha)]\right]$$
$$= \text{Re}\left[\lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha((-s+i\omega)t^\alpha)]\right]$$
$$= \text{Re}\left[\lim_{t\to+\infty}\frac{1}{-s+i\omega}E_\alpha((-s+i\omega)t^\alpha)-\frac{1}{-s+i\omega}\right]$$
$$= \text{Re}\left[\frac{s+i\omega}{s^2+\omega^2}\right] \text{ (since } s>0)$$
$$= \frac{s}{s^2+\omega^2}\ .$$

$$L_\alpha\{sin_\alpha(\omega t^\alpha)\} = ({}_0I^\alpha_{+\infty})[E_\alpha(-st^\alpha)\otimes sin_\alpha(\omega t^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha(-st^\alpha)\otimes sin_\alpha(\omega t^\alpha)]$$
$$= \lim_{t\to+\infty}({}_0I^\alpha_t)\left[\text{Im}[E_\alpha((-s+i\omega)t^\alpha)]\right]$$
$$= \text{Im}\left[\lim_{t\to+\infty}({}_0I^\alpha_t)[E_\alpha((-s+i\omega)t^\alpha)]\right]$$
$$= \text{Im}\left[\lim_{t\to+\infty}\frac{1}{-s+i\omega}E_\alpha((-s+i\omega)t^\alpha)-\frac{1}{-s+i\omega}\right]$$
$$= \text{Im}\left[\frac{s+i\omega}{s^2+\omega^2}\right] \text{ (since } s>0)$$

$$= \frac{\omega}{s^2+\omega^2} \ .$$

$$L_\alpha\{cosh_\alpha(pt^\alpha)\} = \left({}_0I_{+\infty}^\alpha \right)[\,E_\alpha(-st^\alpha) \otimes cosh_\alpha(pt^\alpha)]$$

$$= \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)[E_\alpha(-st^\alpha) \otimes cosh_\alpha(pt^\alpha)]$$

$$= \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)\left[E_\alpha(-st^\alpha) \otimes \frac{1}{2}\left(E_\alpha(pt^\alpha) + E_\alpha(-pt^\alpha)\right)\right]$$

$$= \frac{1}{2} \cdot \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)[(E_\alpha((-s+p)t^\alpha) + E_\alpha((-s-p)t^\alpha))]$$

$$= \frac{1}{2} \cdot \lim_{t\to+\infty} \left[\frac{1}{-s+p} E_\alpha((-s+p)t^\alpha) - \frac{1}{-s+p} + \frac{1}{-s-p} E_\alpha\left((-s-p)t^\alpha\right) - \frac{1}{-s-p}\right]$$

$$= \frac{1}{2} \cdot \left[-\frac{1}{-s+p} - \frac{1}{-s-p}\right] \ \text{(since} > |p|\ )$$

$$= \frac{s}{s^2-p^2} \, .$$

$$L_\alpha\{sinh_\alpha(pt^\alpha)\} = \left({}_0I_{+\infty}^\alpha \right)[\,E_\alpha(-st^\alpha) \otimes sinh_\alpha(pt^\alpha)]$$

$$= \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)[E_\alpha(-st^\alpha) \otimes sinh_\alpha(pt^\alpha)]$$

$$= \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)\left[E_\alpha(-st^\alpha) \otimes \frac{1}{2}\left(E_\alpha(pt^\alpha) - E_\alpha(-pt^\alpha)\right)\right]$$

$$= \frac{1}{2} \cdot \lim_{t\to+\infty} \left({}_0I_t^\alpha \right)[(E_\alpha((-s+p)t^\alpha) - E_\alpha((-s-p)t^\alpha))]$$

$$= \frac{1}{2} \cdot \lim_{t\to+\infty} \left[\frac{1}{-s+p} E_\alpha((-s+p)t^\alpha) - \frac{1}{-s+p} - \frac{1}{-s-p} E_\alpha\left((-s-p)t^\alpha\right) + \frac{1}{-s-p}\right]$$

$$= \frac{1}{2} \cdot \left[-\frac{1}{-s+p} + \frac{1}{-s-p}\right] \ \text{(since} > |p|\ )$$

$$= \frac{p}{s^2-p^2} \, .$$

By first shifting theorem for fractional Laplace transform, we have

$$L_\alpha\{E_\alpha(pt^\alpha) \otimes cos_\alpha(\omega t^\alpha)\} = \frac{s-p}{(s-p)^2+\omega^2} \, ,$$

and

$$L_\alpha\{E_\alpha(pt^\alpha) \otimes sin_\alpha(\omega t^\alpha)\} = \frac{\omega}{(s-p)^2+\omega^2} \, .$$

Q.e.d.

IV. CONCLUSION

The purpose of this article is to find the fractional Laplace transforms of several fractional analytic functions. In fact, these results we obtained are generalizations of Laplace transform of analytic functions. The method used in this paper is also similar to that used in classical Laplace transform. Moreover, the new multiplication we defined is a natural operation in fractional calculus, and it plays a vital role in this paper. In the future, we will also use Jumarie's modified R-L fractional derivative to solve the problems in fractional calculus and applied science.

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